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CITATION:

Bang, Sejeong. A parameter bound on distance-biregular graphs (Algebraic Combinatorics). 数理解析研究所講究録 2005, 1440: 24-27

ISSUE DATE:

2005-07

URL:

<http://hdl.handle.net/2433/47543>

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A parameter bound on distance-biregular graphs

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Abstract

In this note we consider distance-biregular graphs with valencies $(\sigma + 1, \delta + 1)$ with $\sigma < \delta$. We show that bigger valency δ is bounded by a function of smaller valency σ .

Definition 0.1 Let $\Gamma = (V(\Gamma), E(\Gamma))$ be a bipartite connected graph with the vertex set $V(\Gamma)$ and the edge set $E(\Gamma)$. Let D be a diameter of Γ . Then Γ is called a distance-biregular if for any pair of vertices $u, v \in V(\Gamma)$ with $d(u, v) = i$, the numbers $c_i(u, v)$ and $b_i(u, v)$ are depend only on $d(u, v) = i$ and the partition the vertex u belongs to, where

$$\begin{aligned} c_i(u, v) &:= |\Gamma_{i-1}(u) \cap \Gamma_1(v)|, \\ b_i(u, v) &:= |\Gamma_{i+1}(u) \cap \Gamma_1(v)|. \end{aligned}$$

Remark 0.2 (i) The vertices in the same partition have the same intersection array. From now on, let $V(\Gamma) = V_\sigma \cup V_\delta$, where $V_\sigma = \{v \in V(\Gamma) \mid \deg(v) = \sigma + 1\}$ and $V_\delta = \{v \in V(\Gamma) \mid \deg(v) = \delta + 1\}$. (ii) For $u \in V_\sigma$ and $d(u, v) = i$, define

$$\begin{aligned} D_\sigma &:= \max\{d(u, x) \mid x \in V(\Gamma)\} \\ c_i^\sigma &:= |\Gamma_{i-1}(u) \cap \Gamma_1(v)| \\ b_i^\sigma &:= |\Gamma_{i+1}(u) \cap \Gamma_1(v)|. \end{aligned}$$

In the same way, we could define D_δ , c_i^δ and b_i^δ .

Example 0.3 Bipartite distance-regular graphs are distance-biregular graphs.

Lemma 0.4 (Delorme)

For each i ,

- (i) $c_{2i}^\sigma c_{2i+1}^\sigma = c_{2i}^\delta c_{2i+1}^\delta$
- (ii) $b_{2i-1}^\sigma b_{2i}^\sigma = b_{2i-1}^\delta b_{2i}^\delta$.

*This note is based on a joint work with Akira Hiraki(Division of Mathematical Sciences, Osaka Kyoiku University, Japan) and Jack Koolen(Department of Mathematics, Pohang University of Science and Technology, South Korea)

Proof: (i) Let $d(x, y) = 2i + 1$ and $x \in V_\sigma$. By counting the number of paths between x and y in two ways, we have

$$c_{2i+1}^\sigma c_{2i}^\sigma \cdots c_2^\sigma = c_{2i+1}^\delta c_{2i}^\delta \cdots c_2^\delta.$$

(ii) Consider $|\{(u, w) \in V_\sigma \times V_\delta \mid d(u, w) = 2i + 1\}|$. Then

$$\frac{v_\sigma b_0^\sigma b_1^\sigma \cdots b_{2i}^\sigma}{c_2^\sigma \cdots c_{2i}^\sigma c_{2i+1}^\sigma} = \frac{v_\delta b_0^\delta b_1^\delta \cdots b_{2i}^\delta}{c_2^\delta \cdots c_{2i}^\delta c_{2i+1}^\delta}.$$

■

Proposition 0.5 *Let Γ be a distance-biregular graph with valencies $(\sigma + 1, \delta + 1)$. If $\sigma = \delta$ then Γ is distance-regular.*

Proof: By Lemma 0.4, for each $i \geq 1$, we have the following

$$\begin{aligned} (\delta + 1 - c_{2i-1}^\sigma)(\sigma + 1 - c_{2i}^\sigma) &= (\sigma + 1 - c_{2i-1}^\delta)(\delta + 1 - c_{2i}^\delta); \\ c_{2i}^\sigma c_{2i+1}^\sigma &= c_{2i}^\delta c_{2i+1}^\delta. \end{aligned}$$

As $\sigma = \delta$ and $c_1^\sigma = 1 = c_1^\delta$ hold, $c_i^\delta = c_i^\sigma$ ($1 \leq i \leq D$) and $b_i^\delta = b_i^\sigma$ ($0 \leq i \leq D - 1$) must be satisfied. ■

By the previous proposition, we may assume that $\sigma < \delta$ in this note.

Lemma 0.6 (*Delorme*)

Let $\sigma < \delta$. Then

- (i) D_σ is even ;
- (ii) $D_\delta \leq D_\sigma \leq D_\delta + 1$.

Let $g = 2n + 2$ be the girth of a distance biregular graph Γ . Note that $c_{n+1}^\sigma > c_n^\delta = 1$ and $c_{n+1}^\delta > c_n^\sigma = 1$ are satisfied.

Theorem 0.7 (*Delorme*)

$$\max\{D_\sigma, D_\delta\} \leq \frac{g}{2} \min\{\sigma, \delta\} + 1.$$

Proof:

$$\begin{aligned} \max\{D_\sigma, D_\delta\} &\leq \min\{(\sigma - c_{n+1}^\sigma + 2)n + 2, (\delta - c_{n+1}^\delta + 2)n + 2\} \\ &\leq \min\{\sigma, \delta\}n + 2 \\ &\leq \frac{g}{2} \min\{\sigma, \delta\} + 1. \end{aligned}$$

■

We denote by Σ (resp. Δ) the graph with vertices the vertices of degree $\sigma + 1$ (resp. $\delta + 1$) and two vertices are adjacent if and only if they are at distance 2 in Γ . Note that Σ (resp. Δ) is a distance-regular graph of order $(\delta, \frac{\sigma+1}{c_2^\sigma} - 1)$ (resp. $(\sigma, \frac{\delta+1}{c_2^\delta} - 1)$). In other words,

$$\Gamma_1^{(\Sigma)}(x) \simeq \left(\frac{\sigma+1}{c_2^\sigma} \right) * K_\delta.$$

Let N be a matrix whose rows and columns are indexed by $V(\Delta)$ and $V(\Sigma)$ respectively. Then we obtain

$$c_2^\delta A_\Delta = NN^T - (1 + \delta)I; \quad (1)$$

$$c_2^\sigma A_\Sigma = N^T N - (1 + \sigma)I. \quad (2)$$

Lemma 0.8 (*Hoffman-Delsarte*)

Let Γ be a distance-regular graph with diameter D . Let $\theta_0 > \theta_1 > \dots > \theta_D$ be the eigenvalues of Γ . Then the maximal clique in Γ has at most

$$\left(1 - \frac{\theta_0}{\theta_D} \right)$$

vertices.

Theorem 0.9 Let $D_\sigma = D_\delta$. Then every maximal clique in Σ has exactly

$$\left(1 - \frac{\theta_0^{(\Sigma)}}{\theta_{D_\sigma/2}^{(\Sigma)}} \right)$$

vertices.

Theorem 0.10 Let $D_\sigma = D_\delta$. Then every maximal clique in Σ has exactly

$$\left(1 - \frac{\theta_0^{(\Sigma)}}{\theta_{D_\Sigma}^{(\Sigma)}} \right)$$

vertices.

Proof: Let $N \in M_{|V_\delta| \times |V_\sigma|}(\mathbb{R})$. Then

$$N^T N = c_2^\sigma A_{(\Sigma)} + (1 + \sigma)I \in M_{|V_\sigma| \times |V_\sigma|}(\mathbb{R})$$

$$N N^T = c_2^\delta A_{(\Delta)} + (1 + \delta)I \in M_{|V_\delta| \times |V_\delta|}(\mathbb{R}).$$

As $\sigma < \delta$, $|V(\Sigma)| > |V(\Delta)|$. By the fact $\text{rank}(NN^T) = \text{rank}(N) = \text{rank}(N^T N)$ (i.e., the matrix $N^T N$ has an eigenvalue 0) and the equation (2), the graph Σ has an eigenvalue

$$-\frac{1 + \sigma}{c_2^\sigma}.$$

Therefore the following holds:

$$-\frac{1+\sigma}{c_2^\sigma} \leq \theta_{D_\Sigma}^{(\Sigma)} \leq -\frac{1+\sigma}{c_2^\sigma}.$$

Theorem 0.11 (Hiraki and Koolen)

Let Γ be a distance-regular graph of order (s, t) with $s > 1$.

If $-(t+1)$ is an eigenvalue of Γ then

$$t < s^{\frac{4D}{h}-1}.$$

Theorem 0.12 Let Γ be a distance-biregular graph with valencies $\sigma+1$ and $\delta+1$. Let $\sigma < \delta$, $D_\sigma = D_\delta$ and the girth $g \geq 8$. Then there is a function f of σ such that

$$\delta \leq f(\sigma).$$

Proof: Recall that Σ is distance-regular graph of order (σ, δ) . By Theorem 0.11

$$\delta < \sigma^{\frac{4D_\Delta}{h_\Delta}-1}.$$

Now we will show that

$$\frac{4D_\Delta}{h_\Delta} - 1 \leq F(\sigma).$$

Note that the following hold:

$$\begin{aligned} \frac{1}{h_\Delta} &\leq \frac{2}{h-1} = \frac{4}{g-4} \\ D_\Delta &\leq \frac{g\sigma+2}{4}. \end{aligned}$$

Hence

$$\frac{4D_\Delta}{h_\Delta} - 1 \leq \frac{4(g\sigma+2) - g + 4}{g-4} \leq 9\sigma.$$

Therefore $\delta < \sigma^{9\sigma}$ holds.

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